The distribution of extremal points of Gaussian scalar fields

2003 J. Phys. A: Math. Gen. 36 4561

(http://iopscience.iop.org/0305-4470/36/16/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4561–4580 PII: S0305-4470(03)59018-7

The distribution of extremal points of Gaussian scalar fields

Georg Foltin

Institut für Theoretische Physik III, Heinrich–Heine Universität Düsseldorf, Universitätsstrasse 1, D-40225 Düsseldorf, Germany

Received 30 January 2003
Published 8 April 2003
Online at stacks.iop.org/JPhysA/36/4561

Abstract
We consider the signed density of an extremal point of (two-dimensional) scalar fields with a Gaussian distribution. We assign a positive unit charge to the maxima and minima of the function and a negative one to its saddles. At first, we compute the average density for a field in half-space with Dirichlet boundary conditions. Then we calculate the charge–charge correlation function (without boundary). We apply the general results to random waves and random surfaces. Furthermore, we find a generating functional for the two-point function. Its Legendre transform is the integral over the scalar curvature of a four-dimensional Riemannian manifold.

PACS numbers: 05.45.Mt, 03.65.Sq, 02.40.Sf

1. Introduction

In two-dimensional systems with orientational degrees of freedom, it is often fruitful to focus on the nodal points (zeros, defects) and ignore other degrees of freedom. The distribution of the nodal points keeps information about the degree of (orientational) order within the system, and about the topology of the 'substrate'. Well known examples are defects in liquid crystal films, vortices in thin superconductors and superfluid films (for an overview, see [1]), or chaotic wavefunctions in quantum mechanics [2], microwave billiards [3, 4] and optical speckle pattern [5].

Here we take a closer look at the distribution of the nodal points of a gradient field or likewise the extremal points (maxima, minima, saddles) of a scalar field ϕ with a Gaussian distribution[1]. The locations of the extrema of ϕ are the points, where ϕ is stationary, i.e. the gradient $\nabla\phi$ vanishes. The distribution of the stationary points was discussed earlier in the context of random surfaces [6–8] and excursion sets [9, 10].

[1] The distribution of ϕ is solely characterized by the correlation function $\langle\phi(\boldsymbol{r})\phi'(\boldsymbol{r}')\rangle$. Higher moments are calculated with the help of the Wick theorem.

The extremal points carry a topological charge q, which is the sign of the Jacobian right at the extremum $q = \mathrm{sign}(\det(\partial_i\partial_j\phi)) = \pm 1$. In two dimensions, the positive extrema are the maxima and minima of the function ϕ, and the negative extrema are the saddle points. We will compute moments of the *signed* density

$$\rho(\boldsymbol{r}) = \sum_\alpha q_\alpha \delta^2(\boldsymbol{r} - \boldsymbol{r}_\alpha) \tag{1}$$

where $\boldsymbol{r}_\alpha$ are the positions of the extrema with sign (charge) q_α.

From the experimental point of view, measuring the *absolute* mean values of the density, $|\rho|$, is much easier than measuring the usually small density difference between the positive points and the negative ones. The signed density, however, is the much more appealing object from the mathematical point of view. The signed density is subject to a topological constraint, and its averages are differential geometrical objects, as shown in [11].

We represent the signed density of the extremal points of a d-dimensional scalar field ϕ through [7, 8]

$$\rho(\boldsymbol{r}) = \det(\partial_i\partial_j\phi(\boldsymbol{r}))\delta^d(\nabla\phi(\boldsymbol{r})). \tag{2}$$

In order to calculate the average of the absolute density $\langle\rho\rangle$, one needs the simultaneous distribution of the $d(d+1)/2$ random variables $\partial_i\partial_j\phi$ and of the d variables $\nabla\phi$, where both sets are Gaussian random variables. Therefore, we need to know all moments $\langle\partial_i\partial_j\phi\partial_k\partial_l\phi\rangle$, $\langle\partial_i\partial_j\phi\partial_k\phi\rangle$ and $\langle\partial_i\phi\partial_j\phi\rangle$ at coinciding points. For the averaged *signed* density, however, we need less information. According to [11] it is sufficient to know the moments

$$g_{ij}(\boldsymbol{r}) = \langle\partial_i\phi(\boldsymbol{r})\partial_j\phi(\boldsymbol{r})\rangle \tag{3}$$

(which usually depend on the position $\boldsymbol{r}$). The mean density $\langle\rho\rangle$ turns out to be proportional to the (total) curvature of the d-dimensional Riemannian manifold which is described by the above metric tensor g_{ij}. In two dimensions, the mean density is proportional to the Gaussian curvature K of that manifold ('surface') (times the invariant area element $\sqrt{\det g_{ij}}$)

$$2\pi\langle\rho(\boldsymbol{r})\rangle = K\sqrt{\det g_{ij}}. \tag{4}$$

Also, higher order correlation functions fit into this scheme. The correlation function $\langle\rho(\boldsymbol{r}_1)\cdots\rho(\boldsymbol{r}_f)\rangle$ is the total curvature of a $(f \times d)$-dimensional manifold with metric tensor[2]

$$g_{i\alpha,j\beta} = \langle\partial_i\phi(\boldsymbol{r}_\alpha)\partial_j\phi(\boldsymbol{r}_\beta)\rangle \qquad i,j = 1\ldots d \quad \alpha,\beta = 1\ldots f \tag{5}$$

where a pair of a Greek and a Roman index is seen as a single composite index. The charge–charge correlation function for a Gaussian field in two dimensions is therefore the curvature of a particular four-dimensional Riemann manifold. We shall see that the 'Einstein' action, i.e. the covariant integral over the scalar curvature R of the four-dimensional manifold, plays an important role as a generating functional, relating the two-point function and the correlation function of the scalar field ϕ. Before we discuss the two-point function, we present a simple example of a field ϕ with a nonzero mean density $\langle\rho(\boldsymbol{r})\rangle$, namely a system with a straight boundary. The plots of the charge–charge density and the charge density near a wall are somewhat similar, since the latter quantity might be seen as the correlation of a charge with its mirror charge. The general results are applied to random waves, and to thermally fluctuating surfaces (fluid membranes).

[2] We define a $(f \times d)$-dimensional Gaussian scalar function $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_f) = \phi(\boldsymbol{r}_1) + \cdots + \phi(\boldsymbol{r}_f)$. It is obviously $\rho_\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_f) = \rho(\boldsymbol{r}_1) \times \cdots \times \rho(\boldsymbol{r}_f)$, where ρ_Φ is the density of extrema of the field Φ. We obtain the metric tensor $\langle\partial_{i\alpha}\Phi\partial_{j\beta}\Phi\rangle = \langle\partial_i\phi(\boldsymbol{r}_\alpha)\partial_j\phi(\boldsymbol{r}_\beta)\rangle$.

2. Density of extremal points near a boundary

We calculate the signed density of extremal points of a Gaussian distributed random function $\phi(x, y)$ in the upper half-space $y > 0$ and assume Dirichlet boundary conditions $\phi(x, y = 0) = 0$ (see [12]). We begin with a field $\phi(\boldsymbol{r})$ in Euclidean, two-dimensional space with an isotropic and translationally invariant correlation function

$$\langle \phi(x, y)\phi(x', y')\rangle = G\left(\sqrt{(x - x')^2 + (y - y')^2}\right). \tag{6}$$

The correlation function for the upper half-space is obtained by subtracting the appropriate mirror image

$$\begin{aligned} G_+(x, y|x', y') &\equiv \langle \phi(x, y)\phi(x', y')\rangle \\ &= G\left(\sqrt{(x - x')^2 + (y - y')^2}\right) - G\left(\sqrt{(x - x')^2 + (y + y')^2}\right). \end{aligned} \tag{7}$$

The Green function obeys the boundary condition $G_+(x, y|x', y' = 0) = 0$. The components of the metric tensor (3) are easily obtained

$$\begin{aligned} g_{ij}(x, y) &= \langle \partial_i \phi(x, y)\partial_j \phi(x, y)\rangle : \\ g_{xx} &= -G''(0) + \frac{G'(2y)}{2y} \\ g_{yy} &= -G''(0) - G''(2y) \\ g_{xy} &= g_{yx} = 0. \end{aligned} \tag{8}$$

The signed density of extremal points is the scalar curvature of a two-dimensional manifold with metric tensor (8)

$$4\pi\rho(y) = \sqrt{\det g}\, R \tag{9}$$

where the scalar curvature is twice the Gaussian curvature $R = 2K$ in two dimensions. The metric is orthogonal and depends on the distance to the wall y only. The nonvanishing affine connections are

$$\langle \partial_x\partial_y\phi\partial_x\phi\rangle = \partial_y g_{xx}/2 \qquad \langle \partial_y\partial_y\phi\partial_y\phi\rangle = \partial_y g_{yy}/2 \qquad \langle \partial_x\partial_x\phi\partial_y\phi\rangle = -\partial_y g_{xx}/2 \tag{10}$$

since $\langle \partial_x\phi\partial_y\phi\rangle = 0$ and, therefore, $\langle \partial_x\partial_x\phi\partial_y\phi\rangle + \langle \partial_y\partial_x\phi\partial_x\phi\rangle = 0$. The scalar curvature is according to [11]

$$\begin{aligned} R &= (\det g)^{-1} e_{ij} e_{kl} (\langle \partial_i\partial_k\phi\partial_j\partial_l\phi\rangle - \langle \partial_i\partial_k\phi\partial_m\phi\rangle g^{mn} \langle \partial_n\phi\partial_j\partial_l\phi\rangle) \\ &= \frac{2}{g_{xx}g_{yy}}\left(-\frac{1}{2}\partial_y^2 g_{xx} + \frac{1}{4}\partial_y g_{xx}((g_{yy})^{-1}\partial_y g_{yy} + (g_{xx})^{-1}\partial_y g_{xx})\right) \end{aligned} \tag{11}$$

where $e_{12} = 1, e_{21} = -1, e_{11} = e_{22} = 0$. A straightforward calculation yields

$$4\pi\rho(y) = -\partial_y((g_{xx}g_{yy})^{-1/2}\partial_y g_{xx}) = \partial_y f(y) \tag{12}$$

where $f(y) = -(g_{xx}g_{yy})^{-1/2}\partial_y g_{xx}$ is the integrated charge density.

2.1. Random waves in half-space

Chaotic wavefunctions are well modelled by a superposition of partial waves with Gaussian distributed amplitudes and fixed energy [13]. The partial waves ϕ_k must obey the Helmholtz equation $-\nabla^2\phi_k = k^2\phi_k$ with k being fixed. Without loss of generality, we can set $k^2 = 1$. For time reversal symmetry, i.e. in the absence of a magnetic field, we can choose real amplitudes. The corresponding free-space correlation function of ϕ reads [12–17]

$$G(r) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2 p\,\delta(p^2 - 1)\exp(\mathrm{i}\boldsymbol{pr}) \propto J_0(r) \tag{13}$$

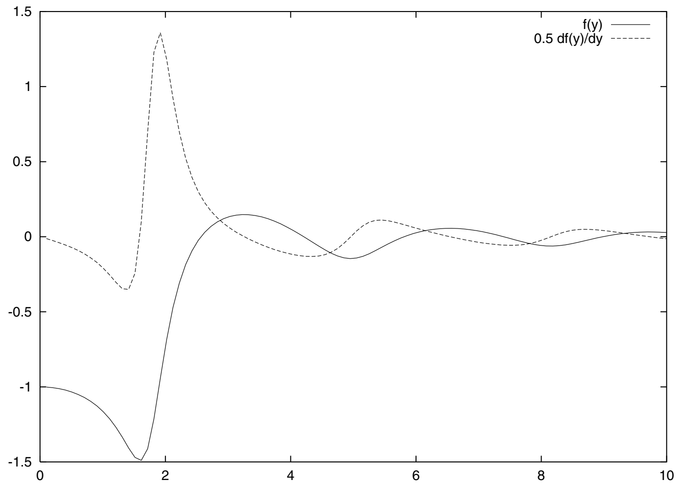

Figure 1. Integrated charge density $f(y)$ (solid) and charge density $2\pi\rho$ (dashed) of a chaotic wavefunction next to a wall.

where J_0 is the zeroth-order Bessel function. $G(r)$ also models the correlation of the undulations $\phi(\boldsymbol{r})$ of capillary waves (with fixed energy) at the surface of a liquid. We might therefore also apply the following results to the extremal points of random capillary waves or even to ocean waves [7, 8, 18–20].

We compute the density of extrema of a random wave near a straight wall with Dirichlet boundary conditions. We expect it to be a model for the distribution of extremal points of a chaotic wavefunction near the boundary [12]. According to section 2 we have to compute the nonvanishing components of the metric tensor (3)

$$\begin{aligned} g_{xx} &= 1/2 - \frac{J_1(2y)}{2y} = (1 - J_0(2y) - J_2(2y))/2 \\ g_{yy} &= 1/2 + J_1'(2y) = (1 + J_0(2y) - J_2(2y))/2 \end{aligned} \tag{14}$$

which can be found in [12]. We obtain $\partial_y g_{xx} = 2J_2(2y)/(2y)$ and the integrated charge density (12)

$$f(y) = \frac{-\partial_y g_{xx}}{\sqrt{g_{xx}g_{yy}}} = \frac{-4J_2(2y)}{2y((1 - J_2(2y))^2 - (J_0(2y))^2)^{1/2}} = -(1 + y^2/8 + \cdots). \tag{15}$$

Figure 1 shows $f(y)$ and its derivative $2\pi\rho = \partial_y f(y)/2$. Next to the wall, one observes a negative density, i.e. an excess of saddle points, followed by a sharp positive peak. The system has, contrary to the free one, a net excess charge $4\pi \int \mathrm{d}y\rho = f(0) = 1/2$ per unit length.

2.2. *A surface of revolution*

There is a good way to visualize the charge density (12). It is possible to embed the manifold with metric tensor (8) in three-dimensional space. This allows us to literally see the defect density, which is the curvature of the embedded surface, times the element of area $\sqrt{\det g}$. There is, however, a price to be paid—we must condense the dimension x by making it 2π-periodic. Then one can construct a surface of revolution with metric tensor (8). The surface

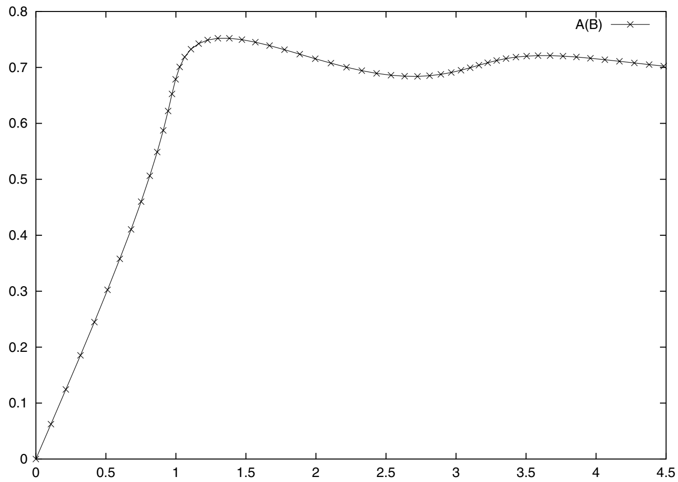

Figure 2. Contour of the surface of revolution. It shows $A(B)$ for $0 \leqslant y < 7$. Two consecutive crosses correspond to an interval of $\Delta y = 1/8$.

of revolution is parametrized through a coordinate y along the principal axis and the angle $0 < x \leqslant 2\pi$,

$$\vec{X}(x, y) = \begin{pmatrix} A(y)\cos x \\ A(y)\sin x \\ B(y) \end{pmatrix} \tag{16}$$

with $A > 0$ and a monotonously growing $B(y)$. $\vec{X}$ is a vector in three-dimensional, Euclidean space. For a compact overview of the differential geometry of surfaces, especially of embedded surfaces, see e.g. [21, 22]. The (induced) metric tensor of the surface of revolution reads

$$\begin{aligned} g_{xx} &= \partial_x \vec{X} \cdot \partial_x \vec{X} = A^2(y) \\ g_{xy} &= 0 \\ g_{yy} &= \partial_y \vec{X} \cdot \partial_y \vec{X} = (\partial_y A)^2 + (\partial_y B)^2. \end{aligned} \tag{17}$$

Thus, the above metric is equal to the metric (8) provided we identify

$$\begin{aligned} A(y) &= \sqrt{g_{xx}} \\ \partial_y B(y) &= (g_{yy} - (\partial_y A)^2)^{1/2} = \sqrt{g_{yy}}\sqrt{1 - f^2(y)/4} \\ B(y) &= \int_0^y \mathrm{d}\bar{y} \sqrt{g_{yy}(\bar{y})(1 - f^2(\bar{y})/4)}. \end{aligned} \tag{18}$$

It is important to keep the information about the coordinate y, e.g. by displaying a grid with lines at equidistant intervals in y-space to be able to reconstruct the element of area $\sqrt{\det g}$. The embedding (16) of the metric tensor (3) is restricted to regions, where the integrated charge density obeys $|f(y)| < 2$. The contour of the surface of revolution for the case of random waves is shown in figure 2, where the function $A(B)$ for $0 \leqslant y < 7$ is displayed. The crosses display the y-coordinate—two consecutive crosses represent an interval $\Delta y = 1/8$. Figures 3 and 4 show two projections of the three-dimensional surface. The meridians are at values of $y = k/4, k = 0, 1, 2, \ldots, 0 \leqslant y < 7$. To find $4\pi\rho$ for a given y, find meridian of number $4 \times y$ (counted from the cone), the local Gaussian curvature and multiply it with $\sqrt{\det g}$, which is proportional to the area of the corresponding grid element.

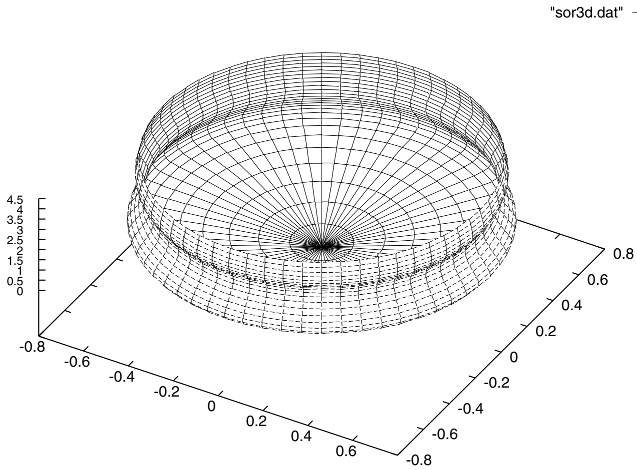

Figure 3. Surface of revolution shown for $0 \leqslant y < 7$. Two consecutive meridian correspond to an interval of $\Delta y = 1/4$.

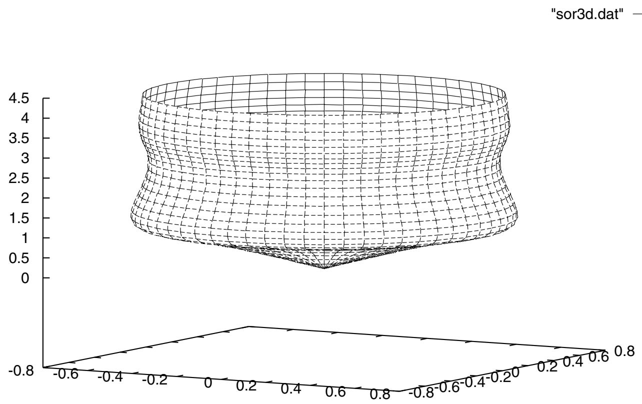

Figure 4. Different view on the surface of revolution shown for $0 \leqslant y < 7$. Two consecutive meridian correspond to an interval of $\Delta y = 1/4$.

The surface of revolution has a conical form for $y \to 0$, which is an artefact of the embedding (16). In fact, expanding the surface for small y we obtain a conical geometry: $A(y) = y/2 + \cdots$ and $B(y) = \sqrt{3}y/2 + \cdots$. The curvature itself is negative at the origin and has the limiting value $R = f'(y)(g_{xx}g_{yy})^{-1/2} \to -1/2$. Figure 5 displays the curvature as a function of the coordinate $R = R(y)$. Note the pronounced positive peak of the scalar curvature at $y \approx 2$.

2.3. Pinned fluctuating surfaces

As a further application, we calculate the density of extrema for a nearly planar, fluctuating fluid membrane, attached to a straight line $y = 0$ (imagine a membrane suspended in a large frame). We represent the shape of the membrane by a height variable $\phi(\boldsymbol{r})$. The domain of $\boldsymbol{r}$ is restricted to the upper half-space $y > 0$, the boundary condition is $\phi(x, y = 0) = 0$. The

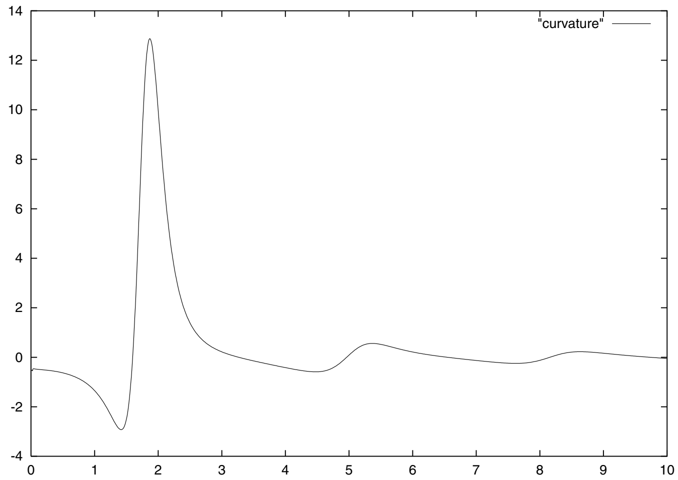

Figure 5. Scalar curvature $R(y)$ for random waves.

thermal shape fluctuations of the fluid membrane are described by the Helfrich–Hamiltonian [23–25]

$$H/T = \int \mathrm{d}^2 r(\kappa(\nabla^2\phi)^2 + \sigma(\nabla\phi)^2) \tag{19}$$

plus higher order terms. κ is the so-called bending rigidity and σ is the effective surface tension. For almost planar surfaces the fluctuations of the height variable ϕ are Gaussian to lowest order. We rescale the height $\sqrt{\kappa}\phi \to \phi$ and set $\tau = \sigma/\kappa$. The correlation function for the unconstrained ϕ is now

$$G(r) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2 p \frac{\exp(\mathrm{i}\boldsymbol{px})}{p^2(p^2+\tau)} = \frac{1}{2\pi}(C + \log(1/r) - K_0(r\sqrt{\tau})) \tag{20}$$

where C is an irrelevant constant. We rescale the length scale that $\tau = 1$ and regularize G at distances $r < a$. With the help of $K_0'(r) = -K_1(r)$ and $K_1'(r) = -(1/2)(K_0(r) + K_2(r))$ and $K_1(r)/r = -(1/2)(K_0(r) - K_2(r))$, we obtain the components of the metric tensor (8)

$$\begin{aligned} g_{xx} &= B - (2y)^{-2} + K_2(2y)/2 - K_0(2y)/2 \\ g_{yy} &= B - (2y)^{-2} + K_2(2y)/2 + K_0(2y)/2 \end{aligned} \tag{21}$$

where $B = -G''(0)$. For distances y much larger than the correlation length (which is 1 in our units), we have $g_{xx} = g_{yy} \approx B - (2y)^{-2}$. The corresponding charge density (12) has an algebraic decay

$$4\pi\rho \sim \frac{3}{2B}y^{-4} \text{ for } r \gg 1 \tag{22}$$

a reminiscence of the long ranged interactions (from the $1/p^2$-term in the Green function). The integrated charge density $f(y)$ is

$$f(y) = \frac{-1/(2y^3) + K_2(2y)/y}{\sqrt{g_{xx}g_{yy}}} = \frac{-1/(2y^3) + K_0(2y)/y + K_1(2y)/y^2}{\sqrt{g_{xx}g_{yy}}}. \tag{23}$$

For small arguments $a < y \ll 1$ the integrated charge density behaves as $f(y) \sim y^{-1}\tilde{F}(\log y) + \mathrm{O}(y^{-2})$. Figure 6 shows $yf(y)$. Figure 7 displays the function $4\pi y^2\rho$, where again y^{-2} is the leading singular behaviour of the charge density for small arguments. The cut-off region $0 < y \lesssim a$ is not shown in both figures.

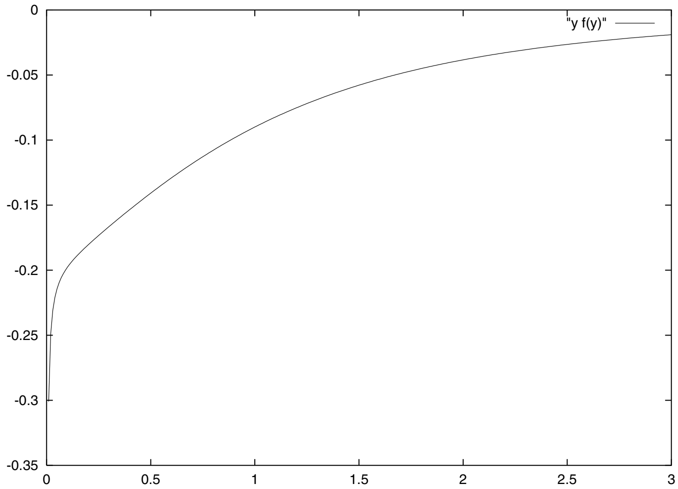

Figure 6. The function $yf(y)$ for membranes.

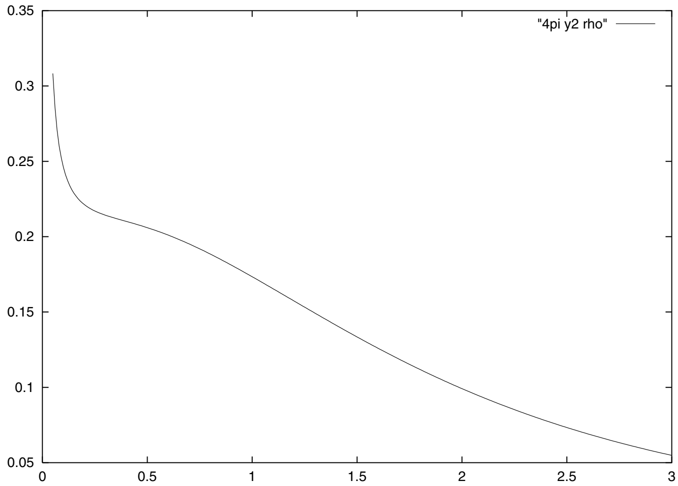

Figure 7. The function $4\pi y^2\rho$ for membranes.

3. The two-point function

We present a convenient way of calculating the two-point function (charge–charge correlation function) $C(r) = \langle\rho(0)\rho(\boldsymbol{r})\rangle$. We have mentioned in the introduction that the two-point function can be expressed through the Riemannian curvature tensor $R_{k\gamma,l\lambda,i\alpha,j\beta}$ of a four-dimensional Riemann manifold with metric tensor $\langle\partial_i\phi(\boldsymbol{r}_\alpha)\partial_j\phi(\boldsymbol{r}_\beta)\rangle$, where $\alpha = 1, 2, \beta = 1, 2$ enumerate the two points. The Riemann curvature tensor is evaluated in appendix B. The derivation of the correlation function with the help of differential geometry is however rather involved. Instead we utilize a less tedious approach, which is even valid for more general Gaussian vector fields $u_i, i = 1, 2$, with an isotropic, translationally invariant correlation function

$$\langle u_i(\boldsymbol{r})u_j(\boldsymbol{r}')\rangle = \chi_{ij}(\boldsymbol{r}-\boldsymbol{r}'). \tag{24}$$

In appendix A we obtain for the charge–charge correlation function

$$(2\pi)^2\langle\rho(\boldsymbol{r})\rho(0)\rangle = (2\pi)^2 C(r) = -\partial_i\partial_j\Omega_{ij} \tag{25}$$

where the potential Ω_{im} reads

$$\Omega_{im}(\boldsymbol{r}) = e_{ij}e_{mn}\tfrac{1}{2}(\det h)^{-1/2}e_{ab}e_{cd}\partial_j\chi_{ac}\partial_n\chi_{bd} \tag{26}$$

with $h_{ij} = \langle u_i u_m\rangle\langle u_j u_m\rangle - \chi_{im}\chi_{jm}$. Summation over double indices is implied. Next, we restrict the vector field u_i to gradient fields $u_i = \partial_i\phi$. The nodal points of u_i, i.e. the points where u_i vanishes, are the extremal point of the scalar field ϕ. The potential becomes

$$\Omega_{im}(\boldsymbol{r}) = e_{ij}e_{mn}\tfrac{1}{2}(\det h)^{-1/2}e_{ab}e_{cd}\partial_j\partial_a\partial_c G\partial_n\partial_b\partial_d G \tag{27}$$

and $h_{ij} = \delta_{ij} - \partial_i\partial_m G\partial_m\partial_j G$. For convenience, we have chosen the amplitude of ϕ that $-G''(0) = 1$. Therefore, it is $-\partial_i\partial_j G(0) = \delta_{ij}$. As a consequence of the polar invariance, matrices (such as $\partial\partial G$, Ω and h) have the general form

$$A_{ij} = A_{11}\frac{r_i r_j}{r^2} + A_{22}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right). \tag{28}$$

For the particular choice $\boldsymbol{r} = (r, 0)$ the only remaining components are A_{11} and A_{22}. The nonvanishing partial derivatives of order 2, 3 and 4 of $G(r)$ and components of h and ξ for $\boldsymbol{r} = (r, 0)$ are given in equation (29). (the prime $'$ denotes derivative with respect to r)

$$\begin{aligned}
&\partial_1\partial_1 G = Z_1 = G''(r)\\
&\partial_2\partial_2 G = Z_2 = G'(r)/r\\
&\partial_1\partial_1\partial_1 G = Z_1'\\
&\partial_1\partial_2\partial_2 G = Z_2' = (Z_1 - Z_2)/r\\
&\partial_1\partial_1\partial_1\partial_1 G = Z_1''\\
&\partial_1\partial_1\partial_2\partial_2 G = Z_2''\\
&\partial_2\partial_2\partial_2\partial_2 G = 3Z_2'/r\\
&h_{11} = D_1 = 1 - Z_1^2\\
&h_{22} = D_2 = 1 - Z_2^2\\
&\xi_{11} = Z_1/D_1\\
&\xi_{22} = Z_2/D_2.
\end{aligned} \tag{29}$$

A short calculation yields

$$\Omega_{11} = -(Z_2')^2(D_1 D_2)^{-1/2} \qquad \Omega_{22} = Z_1' Z_2'(D_1 D_2)^{-1/2} \tag{30}$$

where $Z_1 = G''(r)$, $Z_2 = G'/r$, $D_1 = 1 - Z_1^2$, $D_2 = 1 - Z_2^2$. We find for a matrix of the form (28)

$$\partial_i\partial_j A_{ij} = \nabla^2 A_{11} + \frac{1}{r}\partial_r(A_{11} - A_{22}) \tag{31}$$

where $\nabla^2 A_{11} = (rA_{11}')'/r$. The correlation function becomes

$$(2\pi)^2 C(r) = -\partial_i\partial_j\Omega_{ij} = -\nabla^2\Omega_{11} - (\Omega_{11} - \Omega_{22})'/r \equiv \psi'/r \tag{32}$$

with the 'potential'

$$\begin{aligned}
\psi(r) &= \Omega_{22} - \Omega_{11} - r\Omega_{11}' = \Omega_{22} - (r\Omega_{11})'\\
&= \frac{3(Z_1' - Z_2')(Z_1 - Z_2)}{r(D_1 D_2)^{1/2}} + \frac{(Z_1 - Z_2)^2}{r}((D_1 D_2)^{-1/2})'\\
&= \frac{1}{r(Z_1 - Z_2)}\left(\frac{(Z_1 - Z_2)^3}{(D_1 D_2)^{1/2}}\right)'.
\end{aligned} \tag{33}$$

The correlation function of the Gaussian field ϕ can now be written as

$$G(r) = \int \mathrm{d}p_1\,\mathrm{d}p_2 G\left(\sqrt{p_1^2 + p_2^2}\right)\cos(p_1 r) \equiv \int \mathrm{d}p\tilde{G}(p)\cos(pr) \tag{34}$$

with $G(p) > 0$ and also $\tilde{G}(p) > 0$. It is

$$1 = -G''(r=0) = \int \mathrm{d}p\tilde{G}(p)p^2 \tag{35}$$

i.e. $w(p) = p^2\tilde{G}(p)$ has the properties of a probability density. The r-expansion of $G(r)$ reads

$$G(r) = -\frac{r^2}{2} + \frac{br^4}{4!} - \frac{cr^6}{6!} + \frac{dr^8}{8!} - \frac{er^{10}}{10!} + \mathrm{O}(r^{12}) \tag{36}$$

where $b = G''''(r=0) = \int \mathrm{d}pw(p)p^2$ and $c = -G^{(6)}(r=0) = \int \mathrm{d}pw(p)p^4$ (as well as d and e) are positive constants. With the help of Mathematica™ we find the following series expansion of $\psi(r)$:

$$\psi(r) = \frac{4b}{3\sqrt{3}} + \frac{(b^2-c)r^2}{3\sqrt{3}} + \frac{(45b^4 - 56b^2c + 3c^2 + 10bd)r^4}{540\sqrt{3}b} + \mathrm{O}(r^6). \tag{37}$$

It is shown in equation (40) that $-\psi(0)$ is related to the absolute density of extrema $n_0 = \langle|\rho|\rangle$. The next term of the ψ expansion determines the charge–charge correlation function at almost coinciding points[3]

$$(2\pi)^2 \lim_{r\to 0} C(r) = \frac{-2(c-b^2)}{3\sqrt{3}} = \frac{-2}{3\sqrt{3}}\left(\int \mathrm{d}pw(p)p^4 - \left(\int \mathrm{d}pw(p)p^2\right)^2\right). \tag{38}$$

The latter expression is obviously negative—the vicinity of an extremum is populated by extrema of the opposite type [26, 27], which screen the central charge.

3.1. *Perfect screening of gradient fields*

We show that the charge–charge correlation function $C(r)$ obeys the first Stillinger–Lovett sum rule

$$\lim_{\epsilon\to 0}\int_{|\boldsymbol{r}|>\epsilon} \mathrm{d}^2 r C(r) = -n_0. \tag{39}$$

It expresses the fact that a particular extremum is *completely* screened by extrema (charges) of the opposite sign [15, 28–33]. Applying the results of section 3 we find for the left-hand side

$$-n_0 = 2\pi\int_{0^+}^{\infty} r\,\mathrm{d}r C(r) = \frac{1}{2\pi}\int_{0^+}^{\infty}\mathrm{d}r\psi'(r) = -\frac{1}{2\pi}\psi(0) = -\frac{2G''''(r=0)}{3\pi\sqrt{3}} \tag{40}$$

where $G(r)$ is the correlation function of the field ϕ. On the other hand, the absolute density of extremal points of the Gaussian field ϕ reads (for an arbitrary point $\boldsymbol{r}$)

$$n_0 = \langle|\partial_x\partial_x\phi\partial_y\partial_y\phi - (\partial_x\partial_y\phi)^2|\delta(\partial_x\phi)\delta(\partial_y\phi)\rangle. \tag{41}$$

The Gaussian variables $\partial_i\phi$ are independent of the second derivatives $\partial_i\partial_j\phi$ (evaluated at the same point). It is therefore $n_0 = \langle|\cdots|\rangle\langle\delta(\partial_x\phi)\delta(\partial_y\phi)\rangle$ with

$$\begin{aligned}\langle\delta(\partial_x\phi)\delta(\partial_y\phi)\rangle &= \int \mathrm{d}^2E\frac{1}{\pi\langle(\nabla\phi)^2\rangle}\exp\left(\frac{E^2}{\langle(\nabla\phi)^2\rangle}\right)\delta(E_x)\delta(E_y)\\ &= \frac{1}{\pi\langle(\nabla\phi)^2\rangle} = 1/(2\pi)\end{aligned} \tag{42}$$

[3] $C(0)$ is not well defined, since $C(r)$ has a δ-like singularity at the origin, see [11]. The δ-function, however, is lost 'somewhere' during the course of the derivation 3. A careful analysis at the origin similar to that presented in [11] is needed to recover the δ-function.

where we have chosen $-G''(0) = 1$. The second moments of $\partial_i\partial_j\phi$ at coinciding points are

$$\begin{aligned}
\langle\partial_x\partial_y\phi\partial_x\partial_x\phi\rangle &= \langle\partial_x\partial_y\phi\partial_y\partial_y\phi\rangle = 0 \\
\langle\partial_x\partial_x\phi\partial_x\partial_x\phi\rangle &= \langle\partial_y\partial_y\phi\partial_y\partial_y\phi\rangle = G''''(0) \\
\langle\partial_x\partial_y\phi\partial_x\partial_y\phi\rangle &= \langle\partial_x\partial_x\phi\partial_y\partial_y\phi\rangle = G''''(0)/3.
\end{aligned} \tag{43}$$

Next, we define the mutually independent Gaussian variables $X = (\partial_x\partial_x\phi + \partial_y\partial_y\phi)/\sqrt{2}$, $Y = (\partial_x\partial_x\phi - \partial_y\partial_y\phi)/\sqrt{2}$ and $Z = \partial_x\partial_y\phi$ with $\langle X^2\rangle = (4/3)G''''(0)$, $\langle Y^2\rangle = (2/3)G''''(0)$ and $\langle Z^2\rangle = (1/3)G''''(0)$. Then

$$\begin{aligned}
\langle|\cdots|\rangle &= \langle|X^2/2 - Y^2/2 - Z^2|\rangle \\
&= \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\exp(-(x^2+y^2+z^2)/2)\left|\frac{2G''''(0)}{3}x^2 - \frac{G''''(0)}{3}(y^2+z^2)\right|.
\end{aligned} \tag{44}$$

We introduce spherical coordinates $y = r\sin\theta\cos\varphi$, $z = r\sin\theta\sin\varphi$ and $x = r\cos\theta$, yielding

$$\begin{aligned}
\langle|\cdots|\rangle &= G''''(0)\frac{2}{3\sqrt{2\pi}}\int_0^\infty \mathrm{d}r r^4\exp(-r^2/2)\int_0^{\pi/2}\mathrm{d}\theta\sin\theta|3(\cos\theta)^2 - 1| \\
&= \frac{4G''''(0)}{3\sqrt{3}}.
\end{aligned} \tag{45}$$

Finally, we obtain the absolute density of extremal points of ϕ,

$$n_0 = \frac{2G''''(0)}{3\pi\sqrt{3}} \tag{46}$$

in agreement with equation (40).

3.2. Generating functionals for the two-point function

We have found a representation of the two-point function $C(r)$ as a functional derivative. Although this section is less useful from the practical point of view, it highlights an important mathematical aspect of the theory and demonstrates once more the applicability of the differential geometrical approach. We begin with the integral over the two-point function, times the correlation function of the field ϕ,

$$\begin{aligned}
-2\pi\int \mathrm{d}^2r C(r)G(r) &= (2\pi)^{-1}\int \mathrm{d}^2r\,\partial_i\partial_j\Omega_{ij}G \\
&= \text{b.c.} + (2\pi)^{-1}\int \mathrm{d}^2r\,\Omega_{ij}\partial_i\partial_j G \\
&= \text{b.c.} + \int r\,\mathrm{d}r(\Omega_{11}Z_1 + \Omega_{22}Z_2) \\
&= \text{b.c.} + \int r\,\mathrm{d}r\frac{1}{(D_1D_2)^{1/2}}(Z_1'Z_2'Z_2 - Z_2'Z_2'Z_1) \\
&= \text{b.c.} + \int \mathrm{d}r\frac{Z_1 - Z_2}{(D_1D_2)^{1/2}}(Z_1'Z_2 - Z_2'Z_1) \\
&= -\int \mathrm{d}r\frac{D}{(D_1D_2)^{1/2}}(Z_1' + Z_2') + \int \mathrm{d}r((D_2/D_1)^{1/2}Z_1' + (D_1/D_2)^{1/2}Z_2') \\
&= \mathcal{H} - \mathcal{L}
\end{aligned} \tag{47}$$

where

$$\mathcal{H} = -\int \mathrm{d}r \frac{D}{(D_1 D_2)^{1/2}} (Z_1' + Z_2') \qquad \mathcal{L} = -\int \mathrm{d}r ((D_2/D_1)^{1/2} Z_1' + (D_1/D_2)^{1/2} Z_2') \tag{48}$$

with $D = 1 - Z_1 Z_2$. The action $\mathcal{H}$ is identified in appendix B as the integral over the scalar curvature[4] $R = g^{k\gamma,i\alpha} g^{l\lambda,j\beta} R_{k\gamma,l\lambda,i\alpha,j\beta}$,

$$\mathcal{H} = (4\pi)^{-1} \int \mathrm{d}^2 r \sqrt{\det g}\, R. \tag{49}$$

This functional is well known from general relativity. The curved spacetime manifolds which obey the least-action principle $\delta\mathcal{H} = 0$ are the solutions of the Einstein field equations (without matter). We show below that action (49) is also meaningful for our manifold.

The action $\mathcal{L}$ is the generating functional of $C(r)$ as computed in appendix C,

$$\frac{1}{r} \frac{\delta \mathcal{L}}{\delta G} = \psi'/r = (2\pi)^2 C(r). \tag{50}$$

The Legendre transform of $\mathcal{L}$ reads, using (47) and the above equation,

$$\mathcal{L} - \int \mathrm{d}^2 r \frac{\delta \mathcal{L}}{\delta G(\boldsymbol{r})} G(\boldsymbol{r}) = \mathcal{L} - \int \mathrm{d}r \frac{\delta \mathcal{L}}{\delta G(r)} G(r) = \mathcal{H}. \tag{51}$$

The 'Einstein' action $\mathcal{H}$, expressed as a functional of the charge–charge correlation function $C(r)$, is therefore the generating functional of the conjugated field–field correlation function $G(r)$,

$$\frac{1}{(2\pi)^2 r} \frac{\delta \mathcal{H}[C]}{\delta C(r)} = -G(r). \tag{52}$$

3.3. Two-point function for random waves

We now compute as an application the two-point function $C(r)$ for Gaussian random waves ϕ (without boundary). We rescale the correlation function (13) of the field ϕ for practical reasons and use $G(r) = \langle \phi(\boldsymbol{r})\phi(0)\rangle = 2J_0(r)$. It is $-G''(0) = 1, Z_1 = G''(r) = -J_0(r) + J_2(r)$, $Z_2 = -(J_0(r) + J_2(r))$, $D_1 = 1 - (J_0(r) - J_2(r))^2$, $D_2 = 1 - (J_0(r) + J_2(r))^2$, yielding for the potential (33)

$$\psi(r) = \frac{4}{r J_2(r)} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{(J_2)^3}{(1 - (J_0 - J_2)^2)^{1/2} (1 - (J_0 + J_2)^2)^{1/2}} \right). \tag{53}$$

With the help of the relations $J_0' = -J_1$, $J_2' = J_1 - 2J_2/r$ and $2J_1/r = J_0 + J_2$, we obtain the result

$$\psi(r) = \frac{2(J_2)^3}{(1 - (J_0 - J_2)^2)^{1/2} (1 - (J_0 + J_2)^2)^{1/2}} \left(\left(1 + \frac{J_0}{J_2}\right) \left(\frac{3}{J_2} + \frac{2(J_2 - J_0)}{1 - (J_2 - J_0)^2} \right) \right.$$
$$\left. - \frac{4}{r^2} \left(\frac{3}{J_2} + \frac{J_2 - J_0}{1 - (J_2 - J_0)^2} + \frac{J_2 + J_0}{1 - (J_2 + J_0)^2} \right) \right). \tag{54}$$

The small argument expansion of ψ reads

$$\psi(r) = \frac{1}{\sqrt{3}} \left(1 - \frac{r^2}{48} - \frac{r^4}{2304} - \frac{139 r^6}{29\,859\,840} \right) + \mathrm{O}(r^8). \tag{55}$$

The small r behaviour of the charge–charge correlation function (32) is therefore

$$(2\pi)^2 C(r) = -\frac{1}{24\sqrt{3}} \left(1 + \frac{r^2}{24} + \frac{139 r^4}{207\,360} \right) + \mathrm{O}(r^6). \tag{56}$$

[4] In principle, one has to integrate over four coordinates. The metric, however, depends on the distance $|\boldsymbol{r}^A - \boldsymbol{r}^B|$ only. The integration over the centre of mass is therefore trivial.

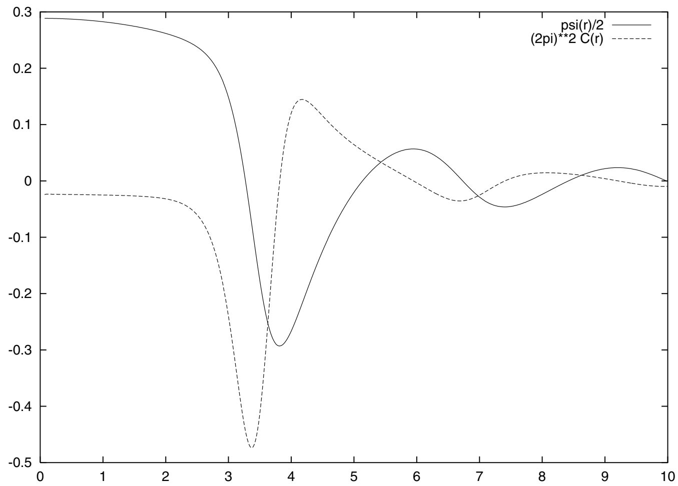

Figure 8. The charge–charge correlation function $(2\pi)^2C(r)$ (dashed) and its potential $\psi(r)$ (solid; $\psi/2$ is shown for scaling reasons).

Figure 8 shows $\psi(r)/2$ and $(2\pi)^2C(r)$ for $0 < r \leqslant 10$. The plateau of the correlation function for small radii is almost perfect due to the smallness of the higher order expansion coefficients. Note the pronounced, negative peak at $r \approx 3.4$.

4. Conclusion

We have studied the signed density of extremal points of a scalar field with a Gaussian distribution. We have calculated the average signed density in half-space with Dirichlet boundary conditions. The density is expressed as the Gaussian curvature of an associated (abstract) manifold, depending on the correlations of the scalar field. The abstract manifold could be visualized by embedding it into three-dimensional space. Then we computed the two-point correlation function of the above density, which is in principle the total curvature of a four-dimensional manifold. We identified the 'Einstein' action as the generating functional, which relates the correlation function of the scalar field and the correlation function of the density of extrema. Currently, no application of this equation is known. Nevertheless, we hope that its further study will ultimately lead to a deeper understanding of the differential geometry of the abstract manifold, which governs the distribution of the extremal points.

Acknowledgments

I would like to thank H K Janssen and U Smilansky for encouraging discussions. This work has been supported by the Deutsche Forschungsgemeinschaft under SFB 237.

Appendix A. Computation of the general charge–charge correlation function

We calculate the charge–charge correlation function for Gaussian vector fields u_i, $i = 1, 2$ with an isotropic, translationally invariant correlation function

$$\langle u_i(\boldsymbol{r}) u_j(\boldsymbol{r}')\rangle = \chi_{ij}(\boldsymbol{r} - \boldsymbol{r}'). \tag{A.1}$$

The charge density ρ is represented as

$$\rho = \frac{1}{2\pi}\partial_i\left(\frac{e_{ij}e_{kl}u_k\partial_j u_l}{u^2+\epsilon}\right) = \frac{1}{2}e_{ij}e_{kl}\partial_i u_k\partial_j u_l\frac{\epsilon}{\pi(u^2+\epsilon)^2}$$
$$\to \det(\partial_i u_j)\delta^2(u) \qquad \text{for} \quad \epsilon\to 0 \tag{A.2}$$

where $e_{12} = 1, e_{21} = -1, e_{11} = e_{22} = 0$. The charge–charge correlation function $C(r) \equiv \langle\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\rangle$ is

$$4\pi^2 C(r) = \frac{\partial}{\partial r_i}\frac{\partial}{\partial r'_m}\left\langle e_{ij}e_{kl}e_{mn}e_{st}\frac{u_k(\boldsymbol{r})\partial_j u_l(\boldsymbol{r})u_s(\boldsymbol{r}')\partial_n u_t(\boldsymbol{r}')}{(u^2(\boldsymbol{r})+\epsilon)(u^2(\boldsymbol{r}')+\epsilon)}\right\rangle$$
$$\equiv -\partial_i\partial_m\Omega_{im}(\boldsymbol{r}-\boldsymbol{r}') \tag{A.3}$$

utilizing the translational invariance of the correlation function (A.1). With the help of the Fourier transformation

$$(u^2+\epsilon)^{-1} = \frac{1}{2\pi}\int \mathrm{d}^2p\,\exp(\mathrm{i}\underline{pu})K_0(p\sqrt{\epsilon}) \tag{A.4}$$

(K_ν is the second modified Bessel function of order ν) we obtain

$$\Omega_{im}(\boldsymbol{r}) = -\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,K_0(\sqrt{\epsilon}p)K_0(\sqrt{\epsilon}q)e_{ij}e_{kl}e_{mn}e_{st}\frac{\partial}{\partial p_k}\frac{\partial}{\partial q_s}\frac{\partial}{\partial A_{jl}}\frac{\partial}{\partial B_{nt}}\bigg|_{A=B=0}$$
$$\times\langle\exp(A_{ij}\partial_i u_j(0) + B_{ij}\partial_i u_j(\boldsymbol{r}) + \mathrm{i}\underline{pu}(0) + \mathrm{i}\underline{qu}(\boldsymbol{r}))\rangle$$
$$= -\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,e_{ij}e_{kl}e_{mn}e_{st}\frac{p_k}{p^2}\frac{q_s}{q^2}\frac{\partial}{\partial A_{jl}}\frac{\partial}{\partial B_{nt}}\bigg|_{A=B=0}$$
$$\times\langle\exp(A_{ij}\partial_i u_j(0) + B_{ij}\partial_i u_j(\boldsymbol{r}) + \mathrm{i}\underline{pu}(0) + \mathrm{i}\underline{qu}(\boldsymbol{r}))\rangle \tag{A.5}$$

in the limit $\epsilon\to 0$. The latter expression was obtained with the help of two partial integrations and $\partial_i K_0(\sqrt{\epsilon}p) = -\sqrt{\epsilon}K_1(\sqrt{\epsilon}p)p_i/p \to -p_i/p^2$. The Gaussian average can now be done, yielding

$$\Omega_{im}(\boldsymbol{r}) = -\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,e_{ij}e_{kl}e_{mn}e_{st}\frac{p_k}{p^2}\frac{q_s}{q^2}\frac{\partial}{\partial A_{jl}}\frac{\partial}{\partial B_{nt}}\bigg|_{A=B=0}$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2 - p_iq_j\langle u_i(0)u_j(\boldsymbol{r})\rangle)$$
$$\times\exp(A_{ij}B_{kl}\langle\partial_i u_j(0)\partial_k u_l(\boldsymbol{r})\rangle + \mathrm{i}p_iB_{kl}\langle u_i(0)\partial_k u_l(\boldsymbol{r})\rangle)$$
$$\times\exp(\mathrm{i}A_{ij}q_k\langle\partial_i u_j(0)u_k(\boldsymbol{r})\rangle + \text{terms} \propto A^2, B^2)$$
$$= -\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,e_{ij}e_{kl}e_{mn}e_{st}\frac{p_k}{p^2}\frac{q_s}{q^2}$$
$$\times(\langle\partial_j u_l(0)\partial_n u_t(\boldsymbol{r})\rangle - p_v\langle u_v(0)\partial_n u_t(\boldsymbol{r})\rangle q_w\langle\partial_j u_l(0)u_w(\boldsymbol{r})\rangle)$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2 - p_iq_j\langle u_i(0)u_j(\boldsymbol{r})\rangle)$$
$$= -\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,e_{ij}e_{kl}e_{mn}e_{st}\frac{p_k}{p^2}\frac{q_s}{q^2}$$
$$\times(-\partial_j\partial_n\chi_{lt}(\boldsymbol{r}) + p_v\partial_n\chi_{vt}(\boldsymbol{r})q_w\partial_j\chi_{lw}(\boldsymbol{r}))$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2 - p_iq_j\chi_{ij}(\boldsymbol{r})) \tag{A.6}$$

where $\langle u_iu_j\rangle = \langle u_i(0)u_j(0)\rangle = \langle u_i(\boldsymbol{r})u_j(\boldsymbol{r})\rangle$. We eliminate the second derivate $\partial_j\partial_n\chi_{lt}$ by a partial 'integration'

$$\Omega_{im}(\boldsymbol{r}) = e_{mn}\partial_n\left(\frac{1}{4\pi^2}\int \mathrm{d}^2p\,\mathrm{d}^2q\,e_{ij}e_{kl}e_{st}\frac{p_k}{p^2}\frac{q_s}{q^2}\partial_j\chi_{lt}(\boldsymbol{r})\exp(\cdots)\right) \tag{A.7}$$

$$+\frac{1}{4\pi^2}\int d^2p\, d^2q e_{ij}e_{kl}e_{mn}e_{st}\frac{p_kp_v}{p^2}\frac{q_sq_w}{q^2}(\partial_j\chi_{lt}\partial_n\chi_{vw}-\partial_n\chi_{vt}\partial_j\chi_{lw})$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2-p_iq_j\chi_{ij}). \tag{A.8}$$

The term (A.7) does not yield a contribution to the charge–charge correlation function $\propto -\partial_i\partial_m\Omega_{im}$ since it is a pure curl. The term $(\partial_j\chi_{lt}\partial_n\chi_{vw}-\partial_j\chi_{lw}\partial_n\chi_{vt})$ is antisymmetric with respect to the indices t, w:

$$\partial_j\chi_{lt}\partial_n\chi_{vw}-\partial_j\chi_{lw}\partial_n\chi_{vt}=e_{tw}e_{ab}\partial_j\chi_{la}\partial_n\chi_{vb}. \tag{A.9}$$

The q^2-denominator is cancelled, since $e_{tw}e_{st}q_sq_w=-q^2$, yielding

$$\Omega_{im}(\boldsymbol{r})\equiv-\frac{1}{4\pi^2}\int d^2p\, d^2q e_{ij}e_{kl}e_{mn}\frac{p_kp_v}{p^2}e_{ab}\partial_j\chi_{la}\partial_n\chi_{vb}$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2-p_iq_j\chi_{ij}) \tag{A.10}$$

where we have omitted the curl (A.7). Now Ω_{im} is no longer symmetric with respect to i, m; nevertheless, only the symmetric part of Ω_{im} contributes to the charge–charge correlation function. Equivalently, we might replace $e_{ab}\partial_j\chi_{la}\partial_n\chi_{vb}$ by the symmetrized version $e_{ab}(\partial_j\chi_{la}\partial_n\chi_{vb}+\partial_n\chi_{la}\partial_j\chi_{vb})/2$. The latter expression is antisymmetric in l, v and can therefore be written as $e_{lv}e_{ab}e_{cd}\partial_j\chi_{ca}\partial_n\chi_{db}/2$. Up to an antisymmetric component, Ω_{im} reads

$$\Omega_{im}(\boldsymbol{r})\equiv e_{ij}e_{mn}\frac{1}{8\pi^2}\int d^2p\, d^2q e_{ab}e_{cd}\partial_j\chi_{ca}\partial_n\chi_{db}$$
$$\times\exp(-(p_ip_j+q_iq_j)\langle u_iu_j\rangle/2-p_iq_j\chi_{ij}). \tag{A.11}$$

Next we perform the p, q-integral, which is Gaussian now, and obtain finally

$$\Omega_{im}(\boldsymbol{r})=e_{ij}e_{mn}\tfrac{1}{2}(\det h)^{-1/2}e_{ab}e_{cd}\partial_j\chi_{ac}\partial_n\chi_{bd} \tag{A.12}$$

where $h_{ij}=\langle u_iu_m\rangle\langle u_ju_m\rangle-\chi_{im}\chi_{jm}$. As an examination, we use equations (A.3) and (A.12) to rederive the well known two-point function for the special case of vector fields with independent components $\chi_{ij}(\boldsymbol{r})=\delta_{ij}G(r)$ in two dimensions (see [11, 15, 30, 32]). Then

$$\Omega_{im}=e_{ij}e_{mn}\frac{1}{G(0)^2-(G(r))^2}\partial_jG\partial_nG=e_{ij}e_{mn}\partial_jK(r)\partial_nK(r) \tag{A.13}$$

where $K(r)=\arcsin(G(r)/G(0))$. Plugged into (A.1), we get

$$4\pi^2C(r)=-e_{ij}e_{mn}\partial_j\partial_mK\partial_i\partial_nK=2\det(\partial_i\partial_jK)=2K''(r)K'(r)/r \tag{A.14}$$

in agreement with the known result.

Appendix B. The 'Einstein' action

We will now calculate the scalar curvature R and the 'Einstein' action $\mathcal{H}=(4\pi)^{-1}\int d^2r\sqrt{\det g}R$ for a four-dimensional Riemannian manifold with metric $g_{i\alpha,j\beta}=-\partial_i\partial_jG(|\boldsymbol{r}^\alpha-\boldsymbol{r}^\beta|)$. In principle, one has to integrate over four coordinates. The metric, however, depends on the distance $|\boldsymbol{r}^A-\boldsymbol{r}^B|$ only. The integration over the centre of mass is therefore trivial. We represent the correlation function of the field ϕ through its Fourier transform

$$\langle\phi(\boldsymbol{r}^A)\phi(\boldsymbol{r}^B)\rangle=G(|\boldsymbol{r}^A-\boldsymbol{r}^B|)=\int d^2p\tilde{G}(p)\exp(i\boldsymbol{p}\cdot(\boldsymbol{r}^A-\boldsymbol{r}^B)). \tag{B.1}$$

The 4×4 metric tensor reads ($i, j\in\{1,2\}$; $\alpha,\beta\in\{A,B\}$)

$$g_{i\alpha,j\beta}=\langle\partial_i\phi(\boldsymbol{r}^\alpha)\partial_j\phi(\boldsymbol{r}^\beta)\rangle=\int d^2p\frac{\partial\psi^*}{\partial r^{i\alpha}}\frac{\partial\psi}{\partial r^{j\beta}} \tag{B.2}$$

where

$$\psi(\boldsymbol{r}^A, \boldsymbol{r}^B; \boldsymbol{p}) = \sqrt{\tilde{G}(p)} \sum_{\alpha=A,B} \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}^\alpha) \tag{B.3}$$

is an immersion of the four-dimensional manifold in function space. Explicitly, we have

$$g_{i\alpha,j\beta} = \delta_{\alpha\beta}\delta_{ij} - c_{\alpha\beta}\partial_i\partial_j G(r) \tag{B.4}$$

where $c_{\alpha\beta} = 1 - \delta_{\alpha\beta}$. The inverse metric tensor is easily calculated with the help of $c_{\alpha\beta}c_{\beta\gamma} = \delta_{\alpha\gamma}$:

$$g^{i\alpha,j\beta} = (\delta_{\alpha\beta}h^{ij} + c_{\alpha\beta}\xi_{ij}) \tag{B.5}$$

where $h_{ij} = \delta_{ij} - \partial_i\partial_m G\partial_m\partial_j G$, $h^{ij} = h^{-1}{}_{ij}$ and $\xi_{ik} = \partial_i\partial_j G h^{jk}$. The determinant of the metric tensor is $\det g = \det h$. The components of the affine connection are

$$\Gamma_{k\gamma;i\alpha,j\beta} = \int \mathrm{d}^2 p\, \partial_{i\alpha}\partial_{j\beta}\psi^*\partial_{k\gamma}\psi = \delta_{\alpha\beta}e_{\alpha\gamma}\partial_i\partial_j\partial_k G(r) \tag{B.6}$$

where the index α is not summed. The components of the Riemannian curvature tensor are calculated from

$$R_{k\gamma,l\lambda,i\alpha,j\beta} = \int \mathrm{d}^2 p(\partial_{i\alpha}\partial_{k\gamma}\psi^*\partial_{j\beta}\partial_{l\lambda}\psi) - \int \mathrm{d}^2 p(\partial_{i\alpha}\partial_{k\gamma}\psi^*\partial_{m\mu}\psi)g^{m\mu,n\nu} \times \int \mathrm{d}^2 p'(\partial_{n\nu}\psi^*\partial_{j\beta}\partial_{l\lambda}\psi) - (i\alpha) \leftrightarrow (j\beta). \tag{B.7}$$

Explicitly, we have

$$\int \mathrm{d}^2 p(\partial_{i\alpha}\partial_{k\gamma}\psi^*\partial_{j\beta}\partial_{l\lambda}\psi) = \delta_{\alpha\gamma}\delta_{\beta\lambda}c_{\alpha\beta}\partial_i\partial_j\partial_k\partial_l G(r) + \delta_{\alpha\beta\gamma\delta}Z_{ijkl} \tag{B.8}$$

and

$$\begin{aligned} R_{k\gamma,l\lambda,i\alpha,j\beta} = {} & (\delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\alpha\lambda}\delta_{\beta\gamma})c_{\alpha\beta}\partial_i\partial_j\partial_k\partial_l G - \delta_{\alpha\gamma}e_{\alpha\mu}\partial_i\partial_k\partial_m G g^{m\mu,n\nu}\delta_{\beta\lambda}e_{\beta\nu}\partial_j\partial_l\partial_n G \\ & + \delta_{\beta\gamma}e_{\beta\mu}\partial_j\partial_k\partial_m G g^{m\mu,n\nu}\delta_{\alpha\lambda}e_{\alpha\nu}\partial_i\partial_l\partial_n G \end{aligned} \tag{B.9}$$

where α and β are not summed, $Z_{ijkl} = \partial_i\partial_j\partial_k\partial_l G|_{r=0}$ and $\delta_{\alpha\beta\gamma\delta} = 1$ if $\alpha = \beta = \gamma = \delta$ and zero otherwise. The Riemannian curvature tensor can be decomposed into

$$R_{k\gamma,l\lambda,i\alpha,j\beta} = e_{\alpha\beta}e_{\gamma\lambda}S_{ijkl} - \delta_{\alpha\beta\gamma\lambda}e_{ij}e_{kl}\omega + c_{\alpha\beta}c_{\gamma\lambda}e_{ij}e_{kl}\Theta/2 \tag{B.10}$$

where we have used $(\delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\beta\gamma}\delta_{\alpha\lambda})c_{\alpha\beta} = c_{\alpha\beta}c_{\gamma\lambda}$ and $(\delta_{\alpha\gamma}\delta_{\beta\lambda} - \delta_{\beta\gamma}\delta_{\alpha\lambda})c_{\alpha\beta} = e_{\alpha\beta}e_{\gamma\lambda}$ (no summation over α, β) and the definitions

$$\begin{aligned} S_{ijkl} &= \partial_i\partial_j\partial_k\partial_l G + (\partial_i\partial_k\partial_p G\partial_j\partial_l\partial_q G + \partial_j\partial_k\partial_p G\partial_i\partial_l\partial_q G)\xi_{pq}/2 \\ \omega &= e_{ij}e_{kl}\partial_i\partial_k\partial_m G\partial_j\partial_l\partial_n G h^{mn}/2 \\ \Theta &= e_{ij}e_{kl}\partial_i\partial_k\partial_m G\partial_j\partial_l\partial_n G\xi_{mn}/2. \end{aligned} \tag{B.11}$$

We obtain easily

$$g^{k\gamma,i\alpha}R_{k\gamma,l\lambda,i\alpha,j\beta} = (\delta_{\beta\lambda}h^{ik} - c_{\beta\lambda}\xi_{ik})S_{ijkl} + \delta_{\beta\lambda}e_{ij}e_{kl}h^{ik}(-\omega + \Theta/2) + c_{\beta\lambda}e_{ij}e_{kl}\xi_{ik}\Theta/2 \tag{B.12}$$

and

$$\begin{aligned} R &= g^{k\gamma,i\alpha}g^{l\lambda,j\beta}R_{k\gamma,l\lambda,i\alpha,j\beta} \\ &= 2\left((h^{ik}h^{jl} - \xi_{ik}\xi_{jl})S_{ijkl} - \frac{2\omega}{\det h} + \frac{\Theta}{\det h} + \Theta\det\xi\right). \end{aligned} \tag{B.13}$$

To simplify this expression, we define

$$S_{ijkl} = T_{ijkl} - (2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\Theta/2 \tag{B.14}$$

where T_{ijkl} can be written as

$$T_{ijkl} = \partial_i\partial_j\partial_k\partial_l G + \partial_i\partial_j\partial_p G\partial_k\partial_l\partial_q G\xi_{pq}. \tag{B.15}$$

The scalar curvature is now

$$R = 2\left((h^{ik}h^{jl} - \xi_{ik}\xi_{jl})T_{ijkl} - \frac{2\omega}{\det h} + \frac{2\Theta}{\det h}\right). \tag{B.16}$$

Furthermore

$$h^{ik}h^{jl}\partial_i\partial_j\partial_m G\partial_k\partial_l\partial_n G\xi_{mn} + 2\Theta/\det h = h^{ij}\partial_i\partial_j\partial_m G h^{kl}\partial_k\partial_l\partial_n G\xi_{mn}. \tag{B.17}$$

For convenience we choose $\boldsymbol{r}^A - \boldsymbol{r}^B = (r, 0)$. The nonvanishing partial derivatives of $G(r)$ for that particular choice of $\boldsymbol{r}$ are listed in equation (29). We have $\omega = Z_1'Z_2'/D_1 - (Z_2')^2/D_2$ and

$$r\frac{\omega}{\sqrt{\det h}} = \frac{1}{\sqrt{D_1D_2}}\left(\frac{Z_1'(Z_1 - Z_2)}{D_1} - \frac{Z_2'(Z_1 - Z_2)}{D_2}\right) = \left(\frac{D}{\sqrt{D_1D_2}}\right)' \tag{B.18}$$

where $D = 1 - Z_1Z_2$. The integral $\int \mathrm{d}^2r(\det h)^{-1/2}\omega$ is therefore zero up to boundary contributions from $r = 0$ and $r \to \infty$. The integral over the scalar curvature $\mathcal{H} = (4\pi)^{-1}\int \mathrm{d}^2r\sqrt{\det g}R$ becomes (up to boundary contributions)

$$\begin{aligned}
\mathcal{H} &= (2\pi)^{-1}\int \mathrm{d}^2r\sqrt{\det h}((h^{ij}h^{kl} - \xi_{ik}\xi_{jl})T_{ijkl}) \\
&= \int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\left(\frac{Z_1''}{D_1} + \frac{2Z_2''D}{D_1D_2} + \frac{3Z_2'}{rD_2} + \left(\frac{Z_1'}{D_1} + \frac{Z_2'}{D_2}\right)^2\frac{Z_1}{D_1}\right. \\
&\quad \left. - (Z_1')^2\left(\frac{Z_1}{D_1}\right)^3 - 3(Z_2')^2\frac{Z_1Z_2^2}{D_1D_2^2}\right) \\
&= \int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\left(\frac{Z_1''}{D_1} + \frac{2Z_2''D}{D_1D_2} + \frac{3Z_2'}{rD_2} + \frac{(Z_1')^2Z_1}{D_1^2} + 2\frac{Z_1'Z_2'Z_1}{D_1^2D_2}\right. \\
&\quad \left. + \frac{(Z_2')^2Z_1}{D_1D_2} - 2(Z_2')^2\frac{Z_1Z_2^2}{D_1D_2^2}\right).
\end{aligned} \tag{B.19}$$

Z_1'' and Z_2'' are eliminated with the help of two partial integrations

$$\int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\frac{Z_1''}{D_1} = \text{b.c.} + \int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\left(-\frac{Z_1'}{rD_1} + \frac{Z_1'Z_2'Z_2}{D_1D_2} - \frac{(Z_1')^2Z_1}{D_1^2}\right) \tag{B.20}$$

$$\begin{aligned}
&2\int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\frac{Z_2''D}{D_1D_2} \\
&\quad = \text{b.c.} + 2\int_0^\infty r\,\mathrm{d}r(D_1D_2)^{1/2}\left(-\frac{Z_2'D}{rD_1D_2} + \frac{Z_1'Z_2'Z_2}{D_1D_2} + \frac{(Z_2')^2Z_1}{D_1D_2}\right. \\
&\quad\quad \left. - DZ_2'\left(\frac{Z_1'Z_1}{D_1^2D_2} + \frac{Z_2'Z_2}{D_1D_2^2}\right)\right)
\end{aligned} \tag{B.21}$$

yielding

$$\mathcal{H} = \text{b.c.} + \int_0^\infty r\,\mathrm{d}r (D_1 D_2)^{1/2} \left(-\frac{Z_1'}{r D_1} - \frac{2 Z_2' D}{r D_1 D_2} + \frac{3 Z_2'}{r D_2} + \frac{3 Z_1' Z_2' Z_2}{D_1 D_2} + \frac{3 (Z_2')^2 Z_1}{D_1 D_2} - \frac{2 (Z_2')^2 Z_2}{D_1 D_2^2} + \frac{2 Z_1' Z_2' Z_1^2 Z_2}{D_1^2 D_2} \right)$$

$$= \text{b.c.} + \int_0^\infty r\,\mathrm{d}r (D_1 D_2)^{1/2} \left(-\frac{Z_1'}{r D_1} - \frac{2 Z_2' D}{r D_1 D_2} + \frac{3 Z_2'}{r D_2} + \frac{Z_1' Z_2' Z_2}{D_1 D_2} + \frac{3 (Z_2')^2 Z_1}{D_1 D_2} - \frac{2 (Z_2')^2 Z_2}{D_1 D_2^2} + \frac{2 Z_1' Z_2' Z_2}{D_1^2 D_2} \right). \tag{B.22}$$

We replace a single $Z_2' = (Z_1 - Z_2)/r$ in the last four terms of the integral

$$\mathcal{H} = \text{b.c.} + \int \mathrm{d}r (D_1 D_2)^{1/2} \left(-\frac{Z_1'}{D_1} - \frac{2 Z_2' D}{D_1 D_2} + \frac{3 Z_2'}{D_2} + \frac{Z_1' (Z_1 - Z_2) Z_2}{D_1 D_2} + \frac{3 Z_2' (Z_1 - Z_2) Z_1}{D_1 D_2} - \frac{2 Z_2' (Z_1 - Z_2) Z_2}{D_1 D_2^2} + \frac{2 Z_1' (Z_1 - Z_2) Z_2}{D_1^2 D_2} \right). \tag{B.23}$$

Identity (B.18) allows us to simplify the last two terms of the above equation

$$2 \int \mathrm{d}r \frac{(Z_1 - Z_2) Z_2}{\sqrt{D_1 D_2}} \left(\frac{Z_1'}{D_1} - \frac{Z_2'}{D_2} \right) = 2 \int \mathrm{d}r\, Z_2 \left(\frac{D}{\sqrt{D_1 D_2}} \right)' = \text{b.c.} - \int \mathrm{d}r \frac{2 Z_2' D}{\sqrt{D_1 D_2}}. \tag{B.24}$$

We gather terms $\propto Z_1'$ and Z_2' and gain the remarkably simple result

$$\mathcal{H} = -\int \mathrm{d}r \frac{D}{\sqrt{D_1 D_2}} (Z_1 + Z_2)' \tag{B.25}$$

up to contributions from the boundary.

Appendix C. Functional derivative of the Lagrangian

We calculate the variation of

$$\mathcal{L} = -\int \mathrm{d}r ((D_2/D_1)^{1/2} Z_1' + (D_1/D_2)^{1/2} Z_2') \tag{C.1}$$

under an infinitesimal variation of the correlation function $G(r) \to G(r) + \eta(r)$. We introduce the shorthand notations $\partial_1 = \partial/\partial Z_1$ and $\partial_2 = \partial/\partial Z_2$ and obtain

$$\delta\mathcal{L} = -\int \mathrm{d}r [((D_2/D_1)^{1/2} \eta''' + (D_1/D_2)^{1/2} (\eta'/r)') + (\partial_1 (D_2/D_1)^{1/2} \eta'' + \partial_2 (D_2/D_1)^{1/2} \eta'/r) Z_1' + (\partial_1 (D_1/D_2)^{1/2} \eta'' + \partial_2 (D_1/D_2)^{1/2} \eta'/r) Z_2'] \tag{C.2}$$

$$= \text{b.c.} + \int \mathrm{d}r [(\partial_1 (D_2/D_1)^{1/2} Z_1' + \partial_2 (D_2/D_1)^{1/2} Z_2') \eta'' + (\partial_1 (D_1/D_2)^{1/2} Z_1' + \partial_2 (D_1/D_2)^{1/2} Z_2') \eta'/r] - \int \mathrm{d}r [(\partial_1 (D_2/D_1)^{1/2} \eta'' + \partial_2 (D_2/D_1)^{1/2} \eta'/r) Z_1' + (\partial_1 (D_1/D_2)^{1/2} \eta'' + \partial_2 (D_1/D_2)^{1/2} \eta'/r) Z_2'] \tag{C.3}$$

$$= \text{b.c.} - \int \mathrm{d}r [(\partial_1 (D_1/D_2)^{1/2} - \partial_2 (D_2/D_1)^{1/2}) (Z_2' \eta'' - Z_1' \eta'/r)]$$

$$= \text{b.c.} + \int \mathrm{d}r\, Y (Z_2' \eta'' - Z_1' \eta'/r) \tag{C.4}$$

where

$$Y \equiv \frac{Z_1 - Z_2}{(D_1 D_2)^{1/2}}. \tag{C.5}$$

The variation becomes

$$\begin{aligned}\delta\mathcal{L} &= \text{b.c.} - \int \mathrm{d}r\,\eta'((YZ_2')' + YZ_1'/r) \\ &= \text{b.c.} + \int \mathrm{d}r\,\frac{\eta'}{r}(2Y(Z_1' - Z_2') + Y'(Z_1 - Z_2))\end{aligned} \tag{C.6}$$

or

$$\frac{1}{r}\frac{\delta\mathcal{L}}{\delta G} = \frac{1}{r}\frac{\partial}{\partial_r}\left(\frac{(Y(Z_1 - Z_2)^2)'}{r(Z_1 - Z_2)}\right) = \psi'/r = (2\pi)^2 C(r). \tag{C.7}$$

References

[1] Chaikin P M and Lubensky T C 1995 *Principles of Condensed Matter Physics* (Cambridge: Cambridge University Press)
[2] Berry M V 1978 Disruption of wavefronts: statistics of dislocations in incoherent Gaussian random waves *J. Phys. A: Math. Gen.* **11** 27–37
[3] Seba P, Kuhl U, Barth M and Stöckmann H-J 1999 Experimental verification of topologically induced vortices inside a billiard *J. Phys. A: Math. Gen.* **32** 8225–30
[4] Barth Michael and Stöckmann Hans-Jürgen 2002 Current and vortex statistics in microwave billiards *Phys. Rev.* E **65** 066208–7
[5] Baranova N B, Zel'dovitch B Ya, Mamaev A V, Pilipetskii N F and Shkukov V V 1981 Dislocations of the wavefront of a speckle-inhomogeneous field (theory and experiment) *JETP Lett.* **33** 195–9
[6] Cartwright D E and Longuet-Higgins M S 1956 The statistical distribution of the maxima of a random function *Proc. R. Soc.* A **237** 212–32
[7] Longuet-Higgins M S 1957 The statistical analysis of a random, moving surface *Phil. Trans. R. Soc.* A **249** 321–87
[8] Longuet-Higgins M S 1957 Statistical properties of an isotropic random surface *Phil. Trans. R. Soc.* A **250** 157–74
[9] Adler Robert J 2000 On excursion sets, tube formulae, and maxima of random fields *Ann. Appl. Probab.* **10** 1–74
[10] Taylor Jonathan E and Adler Robert J 2003 Euler characteristics for Gaussian fields on manifolds *Ann. Probability* **31**
[11] Georg Foltin 2003 Signed zeros of Gaussian vector fields-density, correlation functions and curvature *J. Phys. A: Math. Gen.* **36** 1729
[12] Berry Michael V 2002 Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature *J. Phys. A: Math. Gen.* **35** 3025–38
[13] Berry M V 1977 Regular and irregular semiclassical wavefunctions *J. Phys. A: Math. Gen.* **10** 2083–91
[14] Dennis M R 2001 Topological singularities in wave fields *PhD Thesis* University of Bristol
[15] Berry M V and Dennis M R 2000 Phase singularities in isotropic random waves *Proc. R. Soc.* A **456** 2059–79
[16] Dennis M R 2001 Phase critical point densities in planar isotropic random waves *J. Phys. A: Math. Gen.* **34** L297303
[17] Blum Galya, Gnutzmann Sven and Smilansky Uzy 2002 Nodal domains statistics: a criterion for quantum chaos *Phys. Rev. Lett.* **88** 114101
[18] Longuet-Higgins M S 1960 Reflection and refraction at a random moving surface: I. Pattern and paths of specular points *J. Opt. Soc. Am.* A **50** 838–44
[19] Longuet-Higgins M S 1960 Reflection and refraction at a random moving surface: II. Number of specular points in a Gaussian surface *J. Opt. Soc. Am.* A **50** 845–50
[20] Longuet-Higgins M S 1960 Reflection and refraction at a random moving surface: III. Frequency of twinkling in a Gaussian surface *J. Opt. Soc. Am.* A **50** 851–6
[21] David F 1989 Statistical mechanics of membranes and surfaces *Geometry and Field Theory of Random Surfaces and Membranes* vol 5, ed D R Nelson *et al* (Singapore: World Scientific) 157–223
[22] Kamien Randall D 2002 The geometry of soft materials: a primer *Rev. Mod. Phys.* **74** 953–71

[23] Helfrich W 1973 Elastic properties of lipid bilayers: theory and possible experiments *Z. Naturforsch.* C **28** 693–703

[24] Peliti L and Leibler S 1985 Effects of thermal fluctuations on systems with small surface tension *Phys. Rev. Lett.* **54** 1690–3

[25] Nelson D R, Piran T and Weinberg S (ed) 1989 *Geometry and Field Theory of Random Surfaces and Membranes* vol 5 (Singapore: World Scientific)

[26] Shvartsman Natalya and Freund Isaac 1994 Wave-field phase singularities: near-neighbor correlations and anticorrelations *J. Opt. Soc. Am.* A **11** 2710–8

[27] Freund Isaac and Shvartsman Natalya 1994 Wave-field phase singularities: the sign principle *Phys. Rev.* A **50** 5164–72

[28] Stillinger F and Lovett R 1968 Ion-pair theory of concentrated electrolytes: I. Basic concepts *J. Chem. Phys.* **48** 3858–68

[29] Stillinger F and Lovett R 1968 General restriction on the distribution of ions in electrolytes *J. Chem. Phys.* **49** 1991–4

[30] Halperin B I 1981 Statistical mechanics of topological defects *Physics of Defects* ed R Balian *et al* (New York: North-Holland) pp 813–57

[31] Martin Ph A and Gruber Ch 1983 A new proof of the stillinger-lovett complete shielding condition *J. Stat. Phys.* **31** 691–710

[32] Liu Fong and Mazenko Gene F 1992 Defect–defect correlation in the dynamics of first-order phase transitions *Phys. Rev.* B **46** 5963–71

[33] Freund Isaac and Wilkinson Michael 1998 Critical point screening in random wave fields *J. Opt. Soc. Am.* A **15** 2892–902